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Phenomenology of the $1/N_f$ expansion for field theories in extra dimensions

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Abstract. In this paper we review the properties of the $1/N_f$ expansion in multi-dimensional theories. Contrary to the usual perturbative expansion it is renormalizable and contains only logarithmic divergencies. The price for it is the presence of ghost states, which, however, in certain cases do not contribute to physical amplitudes. In this case the theory is unitary and one can calculate the cross-sections. As an example we consider the differential cross-section of elastic $eq \to eq$ scattering in a $D = 7, 11, \ldots$-dimensional world. We look also for the unification of the gauge couplings in the multi-dimensional standard model and its SUSY extension, which occurs at energies lower than in 4 dimensions.

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1 Introduction

Theories in extra dimensions are one of the best candidates for a new physics beyond the standard model [1–5]. The drawback of these theories is non-renormalizability, since the coupling constant has negative dimension and the perturbative expansion is seek of ultraviolet divergencies. Thus, one can deal with such theories as effective theories [6] valid up to some scale and consider them at the tree level in the hope that some higher energy theory like a string theory will cure all the UV problems.

Recently we have shown that using the $1/N_f$ expansion [7–9] the theory in a multi-dimensional world is renormalizable and has only logarithmic divergencies contrary to the usual perturbative expansion. In this approach the propagators of the gauge fields acquire the improved UV behavior $(p^2)^{1-D/2}$, which has a better convergence in loop integrals. This behavior corresponds to conformal fixed points and leads to logarithmic divergencies in any space time dimension greater than 4. For D odd one has a non-integer power that manifests the idea of unparticle physics proposed recently by Georgi [10, 11].

Effectively the $1/N_f$ expansion leads to the higher-derivative theories that suffer from the presence of ghost states widely discussed in the literature [12–21]. We argued in [7–9] that in some cases these ghost states might be irrelevant, since they do not contribute to the physical amplitudes. We present these arguments below. Provided

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this is true one can consider such amplitudes and calculate the cross-sections to compare them with the ones in 4-dimensional theories. We consider one of such examples.

Several years ago it was suggested that in extra dimensions the running of the couplings is power-like, which leads to the early unification [22, 23]. That was based on the Kaluza–Klein picture of extra dimensions and a special cut-off procedure of higher modes due to the non-renormalizability of the underlying theory. To the contrary, in the case of the $1/N_f$ expansion in extra dimensions one has the renormalizable theory with logarithmic running, however, faster than in 4 dimensions. It leads to the approximate unification both in the Standard Model and in the MSSM.

2 The $1/N_f$ expansion for extra-dimensional theories

Here we briefly discuss the main features of the $1/N_f$ expansion in multi-dimensional space-time [7–9]. As an example we consider the usual QED with N_f fermion fields in D dimensions, where D takes an arbitrary odd value. The Lagrangian looks like

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \bar{\psi}_i(\mathrm{i}\hat{\partial} - m)\psi_i + \frac{e}{\sqrt{N_f}}\bar{\psi}_i\hat{A}\psi_i,$$

(1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

According to the general strategy of the $1/N_f$ expansion [12–17] one first calculates the photon propagator in the leading order of $1/N_f$. Due to transversality of the polarization operator it is useful to apply the Landau gauge. Then in the leading order one has a sequence of bubbles (see Fig. 1) summed up into a geometrical progression. The resulting photon propagator takes the form

$$D_{\mu\nu}(p) = -\frac{\mathrm{i}}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{1}{1 + e^2 f(D)(-p^2)^{D/2-2}},$$
(2)

where

$$f(D) = \frac{\Gamma^2(D/2)\Gamma(2-D/2)}{2^{D-[D/2]-1}\Gamma(D)\pi^{D/2}}$$

and we put $m = 0$ for simplicity.

It useful now to change the normalization of the gauge field $A_\mu \to A_\mu/e$ and introduce the dimensionless coupling h associated with the triple vertex for the interaction of fermions with the gauge field [7–9]. The new coupling h is needed for the multiplicative renormalizability of the theory and provides validity of the renormalization group pole equations [24]. After this, the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}\left(\frac{1}{e^2} + f(D)(\partial^2)^{D/2-2}(1+h)\right)F_{\mu\nu}$$
$$-\frac{1}{2\alpha e^2}(\partial_\mu A_\mu)^2 + \bar\psi_i(\mathrm{i}\hat\partial - m)\psi_i + \frac{\sqrt{h}}{\sqrt{N_f}}\bar\psi_i\hat A\psi_i.$$
(3)

In the non-abelian case in addition one has the triple and quartic self-interactions of the gauge fields. These vertices, which are suppressed by $1/\sqrt{N_f}$ and $1/N_f$, respectively, obtain loop corrections of the same order in $1/N_f$. This leads to extra terms in the effective Lagrangian [7–9].

The Lagrangian (3) is written in a multi-dimensional world. At high energies ($E > M_c$, where M_c is the scale of extra dimensions) the higher-derivative term dominates, while at low energies ($E < M_c$) one has the usual Maxwell term, thus establishing the correspondence with the classical theory. To find the connection to the 4-dimensional world one has to assume some kind of compactification or localization. For example, within the brane-world scenario one may take the fermion fields localized on a 4-dimensional brane and integrate over the extra dimensions. Then one gets the following relation between the D-dimensional and 4-dimensional couplings:

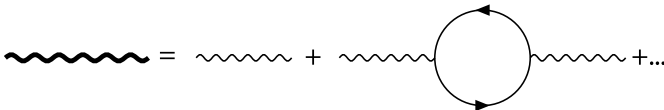$$e_{(D)} = e_{(4)}/V_d, \quad e_{(4)} \equiv \sqrt{h},$$

where $V_d \sim M_c^d$ is the volume of compact extra dimensions or the localization volume. The new dimensionless coupling h enters into the gauge transformation and plays the role of the gauge charge.

As follows from (3) one has the modified Feynman rules with the photon propagator that decreases in the Euclidean region like $1/(p^2)^{D/2-1}$, thus improving the UV behavior in a theory. The only divergent graphs are those of the fermion propagator and the triple vertex. They are both logarithmically divergent for any odd D. The photon propagator is genuinely finite and may contain divergencies only in subgraphs. Therefore, due to the Ward identities in QED, one has a vanishing beta-function and the coupling h is not running. In the non-abelian case this is not true and the beta-function is given by

$$\beta(h) = \frac{-\Gamma(D)(D-1)C_A\frac{h^2}{(1+h)^2}}{2^{[D/2]+2}\Gamma(2-D/2)\Gamma(D/2+1)\Gamma^2(D/2)N_f T}.$$
(4)

One can see that $\beta(h) > 0$ for $D = 5, 9, \ldots$, $\beta(h) < 0$ for $D = 7, 11, \ldots$ and then alternates with D. Hence, one has UV asymptotic freedom for dimensions $D = 7, 11, \ldots$

Consider now the analytical properties of the propagator (2) and the related problem of unitarity. Besides the cut starting from $4m^2$ it has poles in the complex p^2 plane. Hence, knowing the analytical structure, one can write down the Källen–Lehmann representation. It has two contributions: one comes from the pole structure and the other one is the continuous spectrum. Depending on the sign of $f(D)$ there are two possibilities: either one has a pole at the real axis and (possibly) pairs of complex conjugated poles ($f(D) < 0$, $D = 5, 9, \ldots$) or one has only pairs of complex conjugated poles ($f(D) > 0$, $D = 7, 11, \ldots$) and all the rest appears at the second Riemann sheet. In [7–9] it was shown that the continuous spectrum has a positive spectral density and corresponds to the production of pairs of fermions. These states are present in the original spectrum and cause no problem with unitarity. One can show that all the cuts imposed on diagrams when applying the Cutkosky rules in any order of perturbation theory lead to the usual asymptotic states on mass shell, and no new states appear.

The problem comes with the poles. One can see that the pole terms come with a negative sign and, therefore, correspond to the ghost states. For $D = 5$ one has only one pole at the positive real semiaxis, while for $D = 7$ one has a pair of complex conjugated poles, as shown in Fig. 2.
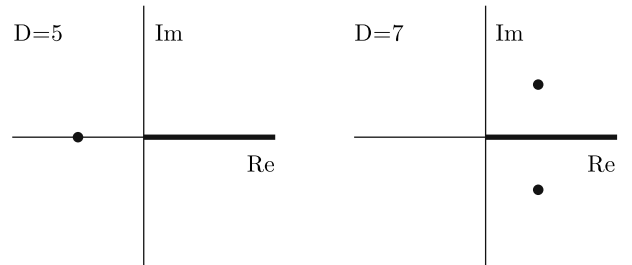

Fig. 1. The chain of diagrams giving a contribution to the photon propagator in the zeroth order of the $1/N_f$ expansion

Fig. 2. The analytical structure of the gauge field propagator

The presence of those ghost states is a drawback of a theory. They signal the instability of the vacuum state. So one either has to try to get rid of the ghost poles or to make sure that they do not give a contribution to the physical amplitudes. It was shown in [7–9] that the contribution of those states to the physical amplitudes is canceled in $D = 7, 11, \ldots$ due to the complex conjugation of the poles in (2) and they do give a contribution to the physical amplitudes in $D = 5, 9, \ldots$

3 The cross-section of elastic scattering

To find the phenomenological consequences of extra-dimensional theories within our approach we calculate the cross-section of elastic $e^- q$ scattering: $e^- q \to e^- q$. In the standard model for this process one has only one diagram in the t-channel (see Fig. 3), which gives the differential cross-section [25]

$$\frac{\mathrm{d}\sigma_{\mathrm{SM}}}{\mathrm{d}t} = \frac{2\pi\alpha^2 Q_f^2}{s^2}\left(\frac{s^2 + u^2}{t^2}\right). \qquad (5)$$

To calculate the same cross-section in a multi-dimensional world we assume the brane-world scenario [26] and consider the scattering of particles that are localized on a 4-dimensional brane while the photon field propagates in the bulk and has the propagator given by (2). The interaction term on the brane now looks like

$$\frac{\sqrt{h}}{\sqrt{N_f}}\int \mathrm{d}^4 x\, \mathrm{d}^d y\, \bar\psi(x)\gamma^\mu A_\mu^a(x,y)\psi(x)\delta^d(y).$$

Given this setup, to get the cross-section one just has to replace the photon propagator in (5) by a modified one and to integrate over the extra dimensions:

$$\frac{1}{t} \Rightarrow \frac{(-)^{d/2}}{(2\pi)^d}\int \frac{\mathrm{d}^d p}{(t - p^2)\left(1/e_{(D)}^2 + N_f f(D)(p^2 - t)^{d/2}\right)}.$$

To compare the cross-section in a multi-dimensional world with the one in the standard model we assume that the extra dimensions appear at some scale M_c: below this scale one has the usual 4-dimensional cross-section and above this scale the full multi-dimensional theory is applied. So one has to merge the two cross-sections at the compactification scale M_c. Working in the center of mass frame and taking $\theta = \pi/2$ one gets $t = -s/2$ and $u = -s/2$. Requiring that

$$\left.\frac{\mathrm{d}\sigma_{\mathrm{SM}}}{\mathrm{d}t}\right|_{s=M_c^2} = \left.\frac{\mathrm{d}\sigma_D}{\mathrm{d}t}\right|_{s=M_c^2}, \qquad (6)$$
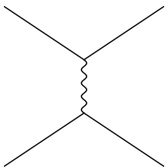

Fig. 3. The $e^- q \to e^- q$ scattering process

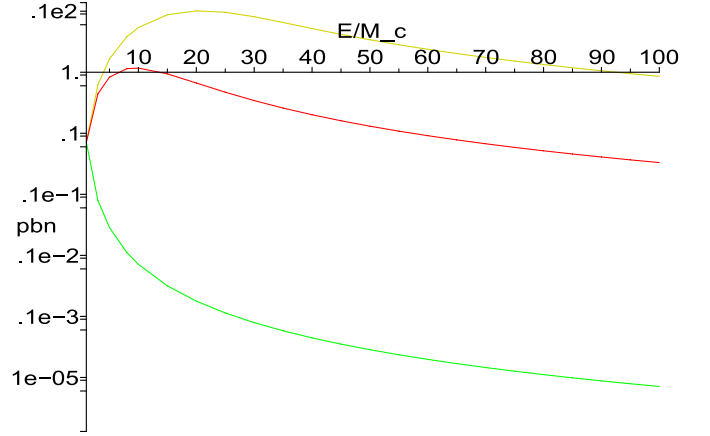**Fig. 4.** The differential cross-section for $eq \to eq$ scattering

one gets the equation for $e_{(D)}^2$. Taking for example $D = 7, 11$ and $N_f = 3$ we find numerically $1/e_{(7)}^2 \simeq 0.2988 M_c^3$, $1/e_{(11)}^2 \simeq 312.96 M_c^7$. Finally we obtain the following expression for the differential cross-section in $D = 4 + d$ dimensions in the c.m.f.:

$$\frac{\mathrm{d}\sigma_D}{\mathrm{d}t} = \frac{5\pi e^2 Q_f^2}{2(2\pi)^{2d}}$$
$$\times \left(\int \frac{\mathrm{d}^d p}{(s/2 + p^2)\left(X M_c^d + 3 f(D)(s/2 + p^2)^{d/2}\right)}\right)^2, \qquad (7)$$

where X is found from (6); it depends on the number of dimensions D.

As an illustration we show in Fig. 4 the resulting cross-section in the case of $D = 7, 11$ for $\theta = \pi/2$. The compactification scale M_c is taken to be equal to 1 TeV. For comparison we also show the 4-dimensional cross-section (5).

One can see that above the compactification scale the cross-sections differ essentially; however, asymptotically they exhibit the same behavior, decreasing like $1/s$ contrary to the Kaluza–Klein approach and in agreement with the fixed point behavior [27].

4 The gauge coupling unification

Here we would like to apply the $1/N_f$ expansion to the problem of gauge coupling unification. As is well known, the gauge coupling unification does not take place in the standard model but can easily be achieved in the MSSM. It occurs at the very high energy scale of the order of 10^{16} GeV, which is far beyond a direct search. At the same time in [22, 23] it was suggested that one can get early unification in theories in extra dimensions due to the power law running of the couplings.

Our approach to extra-dimensional theories based on the $1/N_f$ expansion allows us to look at this problem differently. We get the logarithmic running but with slightly different beta-functions: for the U(1) coupling constant we

have the zero beta-function, which means that the U(1) coupling does not run and for the SU(2) and SU(3) couplings we have the beta-function given by (4). Substituting the proper Casimir operators and taking the number of families $N_F = 3$ one can solve the RG equations numerically and check the unification.

We apply the following procedure: below the unification scale we take the usual RG equations. Then we use the values of the couplings at the compactification scale as initial conditions for the RG equations in the extra dimensions and run the dimensionless couplings $\alpha_i = h_i/4\pi$
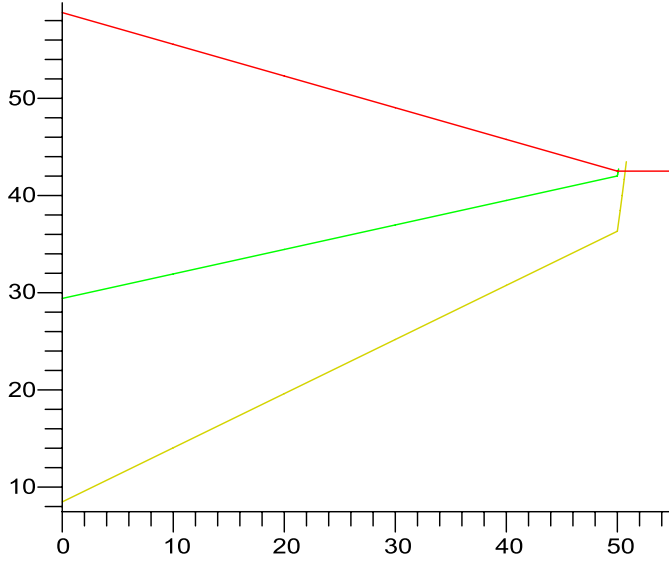
according to the new equations. The compactification scale is varied to get better unification.

The resulting curves for the inverse couplings in the case of the standard model are shown in Fig. 5 for $D = 7$. We would like to stress here that we do not have exact unification at one point but rather the triangle of the unification though much smaller than in the usual 4-dimensional SM. The compactification scale happens to be of the order of $M_c \sim 7 \times 10^{12}$ GeV.

In the MSSM case one has to add the contributions of N_f superpartners; however, the picture does not change much: qualitatively, unlike the 4-dimensional case, one has the same pattern. In Fig. 6 for $D = 7$ we show the plots for two possible choices of the compactification scale: on the left plot we take $M_c \sim 4 \times 10^{10}$ GeV and on the right plot $M_c \sim 7 \times 10^{12}$ GeV. In both cases the SUSY scale is taken to be the same $M_{\mathrm{SUSY}} \sim 500$ GeV. Again one can get approximate unification; the better, the larger the compactification scale. From this point of view we do not see here much difference between the SM and the MSSM.

5 Conclusion

We conclude that within the $1/N_f$ approach the multidimensional theory can be made meaningful, renormalizable and manageable. And though it necessarily contains ghost states in case when they are complex conjugated, their contribution to the physical amplitudes cancels and the theory seems to be unitary.

In such a theory one can calculate the cross-sections of elementary processes assuming a certain compactification (localization) scenario. These cross-sections differ from
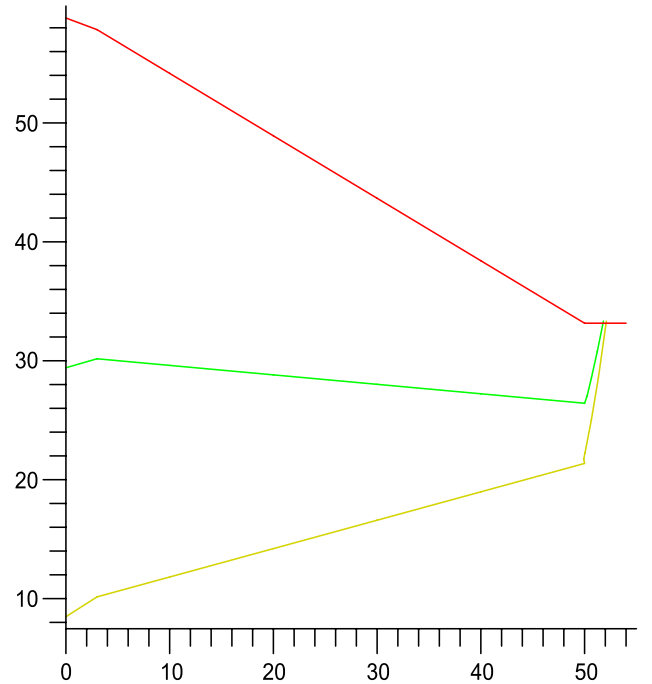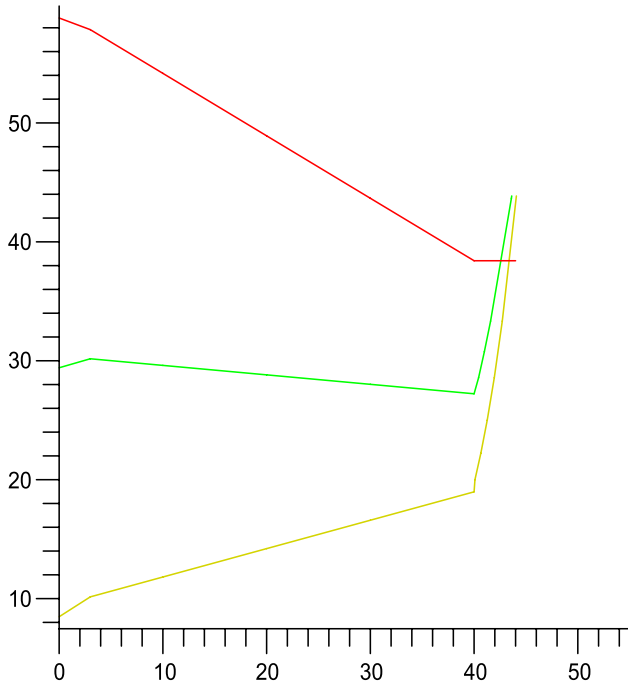

Fig. 5. The running of the coupling constants in the SM

Fig. 6. The running of the coupling constants in the MSSM

those of the standard model in 4 dimensions but asymptotically have the naïve scaling behavior and, hence, do not violate unitarity.

The running of the dimensionless couplings in such theories remains logarithmic and allows one to get an approximate unification of the gauge couplings both in the standard model and in the MSSM, though the unification scale happens to be rather high; yet it is lower than in the usual MSSM.

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